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Vibration and Robust Control of Symmetric Flexible Systems

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The use of symmetry in the vibration analysis and control of symmetric flexible systems is investigated. For a symmetric flexible system with actuators/sensors being symmetrically allocated, an internal and external decoupling control scheme is developed to simplify the controller design. The whole flexible system can be separated into two uncoupled subsystems under the decoupling scheme, and the control algorithm for each subsystem can be designed and implemented independently. Consider the case of a direct output feedback based controller design with actuators and sensors collocated. Two robustness properties of this controller, i.e., full-order closed-loop stability and reliable control, are exploited. For demonstrative purposes, the optimal modal space approach is adopted for controller design of each subsystem and to illustrate the advantages of the decoupling control scheme. Simulation results of the control of a uniform simply supported beam are given to show the effectiveness of this proposed scheme.

Introduction

THE use of symmetry is an important skill in many fields. It can simplify the analysis and design of physical systems as well as add beauty and balance to them. Although many complex physical systems of interest, such as ships, aircraft, and missiles, are not symmetric but have at least approximate symmetry, the developed symmetry methods are still applicable to simplify the analysis to some extent. For this purpose, vibration and control of symmetric flexible systems are studied in this paper.

Vibration analysis of symmetric flexible systems has been discussed by Greenwood[1] and Meirovitch.[2] They observed that, because of the symmetry of the system, the solution of the eigenvalue problem consists of vibration modes of two types, namely, symmetric and antisymmetric with respect to the symmetric center. Greenwood simplified vibration analysis of these systems by first breaking the system into smaller parts and then analyzing each equivalent subsystem. Meirovitch illustrated the simplification for a flexible system with symmetric mass and stiffness properties by using an assumed modes method to separate the original eigenvalue problem into two eigenvalue problems of lower order. However, the description given by Greenwood and Meirovitch did not give a deeper insight into the properties of the system.

The simplification in controller design for a symmetric flexible system is, in essence, an extension to that in vibration analysis. In addition to the system itself being symmetric, if the control components are symmetrically allocated (i.e., the composite system still retains the symmetry property), a great many advantages can be obtained. For example, the enhancement of the robustness stability of the control system is one of the very important results.

The guarantee of stability is almost a basic problem to the controller design. In the area of active control of flexible systems, the inevitable and unwanted spillover effect due to the effect of reduced-order control may make the full-order closed-loop (FOCL) system unstable.[3,4] Although various ways of alleviating or suppressing such an effect have been proposed, the result is finite.[5] Alternatively, instead of eliminating the effect, Salm,[6] using the Kelvin-Tait-Chetaev stability theorem for second-order mechanical systems, presents a principle for the reduced-order controller design that guarantees the FOCL stability by satisfying a sufficient condition. Recently, Juang and Phan[7] also presented a robust controller design for second-order mechanical systems. Conditions on actuator and sensor placements are identified for controller designs that guarantee overall closed-loop stability. The controller is model independent and can be viewed as a virtual passive damping system that serves to stabilize the actual dynamic system. The method is also extended to nonlinear mechanical systems.[8] Another effect that can destroy the stability is the failure of control components. Many methods of performing the failure detection and isolation (FDI) function have been proposed; however, the feasibility of their application to large space structures (LSS) deserves to be assessed since many factors such as spillover effect and uncertainty directly affect the reliability of these methods.[9,10] As an alternative, instead of performing the FDI function, reliable stabilization using a multicontroller configuration[11-13] is developed to ensure safety of the failed system.

In this paper, a more precise and systematic formulation to the symmetric flexible system is given. To express the symmetry property of physical systems in a mathematical description, the symmetric matrix with reflection symmetry is introduced. Then, two theorems stemming from this type of matrix are developed. They result in a great saving of effort in analysis, especially when the system is complex and with large degrees of freedom. When the control components (including actuators and sensors) are symmetrically allocated, an internal and external decoupling scheme is developed. The whole controlled system (including the system equation and output equation) can be separated into two uncoupled subsystems under the decoupling scheme; and, in consequence, the control algorithm can be applied independently. It helps us to reduce the effort in solving control problems, e.g., resolution of the Riccati equation, and implies that for LSS the implementation can be executed by two computers independently. Moreover, when a direct output feedback based controller design with collocation of actuators and sensors is utilized, two robustness properties are discussed. One is the guarantee of FOCL stability in spite of reduced-order control. The other is a reliable control of passive type, which can accommodate the effect of actuators/sensors failure and does not need to perform the FDI function. Finally, for demonstrative purposes, in each subsystem an optimal modal space approach is applied to design the output feedback gain. As an illustration, the control of a uniform simply supported beam is used to show the decoupling controller design and its robustness properties.

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Symmetric Matrix with Reflection Symmetry

The use of quadratic forms is widespread in the fields of geometric analysis and dynamic modeling, e.g., the analysis of the eigenvalue problem and the modeling of small motions of dynamic systems. In essence, the symmetric matrix in the quadratic form plays a central role during both analyses; it describes the geometric interpretation during the analysis of the eigenvalue problem, and it determines the dynamic properties, e.g., mass and stiffness, of the flexible system. First let us consider a quadratic form in n variables denoted by

$$Q(x) = x^T M x \tag{1}$$

where $M = [m_{ij}]$ is a symmetric matrix and $x = [x_1, x_2, \ldots, x_n]^T$.

Now consider a special type of quadratic form for which the central quadratic surface possesses reflection symmetries with respect to the $x = Bx$ surface and $x = (-B)x$ surface simultaneously, where B is the backward identity (see Appendix A). Then the matrix M for such a quadratic form is invariant under reflection transformations B and $-B$. That is to say, for this type of quadratic form, the matrix M not only is symmetric but also satisfies

$$M = B^T M B = BMB \tag{2}$$

and

$$M = (-B)M(-B) \tag{3}$$

In fact, as defined in set theory, this type of symmetry possesses a transformation group that is written as

$$T_g = \{I, B, -B, -I\} \tag{4}$$

where I is the identity.

In many applications, for the sake of convenience, it is desirable to express the quadratic form as a linear combination of only squares of the variables, the cross-product terms being eliminated. A form of this type is said to be a canonical form. Now, based on Eqs. (2) and (3), we bring in a theorem to depict how the original symmetric matrix can be transformed into a block diagonal matrix. Thereby, much computational effort can be saved to obtain this canonical form. Simply put, the resolution of the original eigenvalue problem is reduced to that of two eigenvalue problems of lower order.

Theorem 1: Let M be a $2n \times 2n$ square matrix and suppose that

$$M = B_{2n} M B_{2n} \tag{5}$$

where B_{2n} is a backward identity with dimension $2n \times 2n$. There exists a transformation matrix P denoted by

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{6}$$

satisfying

$$P_{11} = B_n P_{21} \quad \text{and} \quad P_{12}^T = -P_{22}^T B_n \tag{7}$$

or

$$P_{11} = -B_n P_{21} \quad \text{and} \quad P_{12}^T = P_{22}^T B_n \tag{8}$$

so that the matrix M can be transformed into a block diagonal matrix, i.e.,

$$P^T M P = \begin{bmatrix} \bar{M}_{11} & 0 \\ 0 & \bar{M}_{22} \end{bmatrix} \tag{9}$$

where $\bar{M}_{11}$ and $\bar{M}_{22}$ are symmetric if M is symmetric.

Proof: Let M be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{10}$$

Then $P^T M P$ can be rewritten in partitioned form as

$$P^T M P = \begin{bmatrix} P_{11}^T M_{11} P_{11} + P_{11}^T M_{12} P_{21} + P_{21}^T M_{21} P_{11} + P_{21}^T M_{22} P_{21} & P_{11}^T M_{11} P_{12} + P_{11}^T M_{12} P_{22} + P_{21}^T M_{21} P_{12} + P_{21}^T M_{22} P_{22} \\ P_{12}^T M_{11} P_{11} + P_{12}^T M_{12} P_{21} + P_{22}^T M_{21} P_{11} + P_{22}^T M_{22} P_{21} & P_{12}^T M_{11} P_{12} + P_{12}^T M_{12} P_{22} + P_{22}^T M_{21} P_{12} + P_{22}^T M_{22} P_{22} \end{bmatrix} \tag{11}$$

$$P^T M P = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{21} & \bar{M}_{22} \end{bmatrix} \tag{12}$$

Since M possesses the property $M = B_{2n} M B_{2n}$, we have

$$B_n M_{11} B_n = M_{22} \tag{13}$$

$$B_n M_{12} B_n = M_{21} \tag{14}$$

Substituting Eqs. (13), (14), and (7) [or Eq. (8)] into Eq. (12), we immediately obtain

$$\bar{M}_{12} = 0 = \bar{M}_{21} \tag{15}$$

Further, let M be symmetric, and we can obtain

$$\bar{M}_{11}^T = \bar{M}_{11} \quad \text{and} \quad \bar{M}_{22}^T = \bar{M}_{22} \tag{16}$$

□

The advantage of this theorem is that the determination of transformation matrix P is independent of the entries of matrix M. This result suggests that more than one matrix can simultaneously be transformed into block diagonal matrices if the transformed matrices satisfy the property $M = BMB$.

As a matter of fact, the extension of the property to a nonsquare matrix can be validated. In the sequel, a more generalized theorem, in comparison with Theorem 1, is presented.

Theorem 2: Let D be a $2n \times 2m$ nonsquare matrix and suppose that

$$D = B_{2n} D B_{2m} \tag{17}$$

where B_{2n}, B_{2m} is the backward identity matrix. There exists a pair of transformation matrices P and Q denoted by

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}_{2n \times 2n} \tag{18}$$

and

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}_{2m \times 2m} \tag{19}$$

satisfying

$$P_{11} = B_n P_{21} \quad \text{and} \quad P_{12}^T = -P_{22}^T B_n \tag{20}$$

or

$$P_{11} = -B_n P_{21} \quad \text{and} \quad P_{12}^T = P_{22}^T B_n \tag{21}$$

and

$$Q_{11} = B_m Q_{21} \quad \text{and} \quad Q_{12}^T = -Q_{22}^T B_m \tag{22}$$

or

$$Q_{11} = -B_m Q_{21} \quad \text{and} \quad Q_{12}^T = Q_{22}^T B_m \tag{23}$$

so that the matrix D can be transformed into a block diagonal matrix, i.e.,

$$P^T D Q = \begin{bmatrix} \bar{D}_{11} & 0 \\ 0 & \bar{D}_{22} \end{bmatrix} \tag{24}$$

Proof: The results follow similarly from the proof of Theorem 1 and are omitted here.

□

Symmetric Flexible System

The geometric analysis for the symmetric matrix with reflection symmetry is now extended to the vibration analysis of symmetric flexible systems.

First let us consider a conservative flexible system whose equations of motion are denoted by

$$M\ddot{q} + Kq = 0 \tag{25}$$

where the mass matrix M and the stiffness matrix K are constant and symmetric. The q are usually considered to be ordinary coordinates but could be any set of generalized independent coordinates that specify the configuration of the system and are measured from the equilibrium position. Therefore, it is noted that the matrices M and K depend on the set of generalized coordinates used.

Now consider a special type of flexible system in which the mass and stiffness properties M and K possess reflection symmetries with respect to the $q = Bq$ surface and the $q = (-B)q$ surface simultaneously. Then the mass and stiffness properties for such a system are invariant under reflection transformations B and $-B$. That is to say, for this type of system, the matrices M and K not only are symmetric but also satisfy

$$M = B^T M B = BMB \tag{26}$$

$$K = B^T K B = BKB \tag{27}$$

and

$$M = (-B)M(-B) \tag{28}$$

$$K = (-B)K(-B) \tag{29}$$

Having introduced this special type of flexible system, the characteristics are further investigated. Consider the eigenvalue problem

$$(K - \lambda M)V = 0 \tag{30}$$

where λ is the eigenvalue and V is the eigenvector. First use the reflection symmetry property, Eqs. (26) and (27), and substitute them into Eq. (30) to yield

$$(BKB - \lambda BMB)V = 0 \tag{31}$$

Rearranging Eq. (31), we can write

$$(K - \lambda M)BV = 0 \tag{32}$$

It turns out that BV is the eigenvector and possesses the property

$$V = BV \tag{33}$$

At the same time, using another reflection symmetry property, Eqs. (28) and (29), and substituting them into Eq. (30), we can also obtain

$$(K - \lambda M)(-B)V = 0 \tag{34}$$

It shows that the other eigenvector $(-BV)$ exists and possesses the property

$$V = (-B)V \tag{35}$$

From the viewpoint of geometry, the property of Eq. (33) can be called symmetry and that of Eq. (35) can be called antisymmetry. Hence, for this type of flexible system, the solution of the eigenvalue problem consists of eigenvectors of two types, namely, symmetric and antisymmetric with respect to the geometric symmetric center. Representing the symmetric and antisymmetric eigenvectors as V_s and V_a, respectively, their scalar product can be written as

$$(V_s, V_a) = V_s^T V_a = V_s^T B(-B)V_a = -V_s^T V_a = -(V_s, V_a) \tag{36}$$

Since the product (V_s, V_a) is real, it follows that $(V_s, V_a) = 0$. Thus, we deduce that these two types of eigenvectors not only are orthogonal relative to M but also orthogonal relative to I.

Internal and External Decoupling Scheme

The application of the theoretical results of Theorems 1 and 2 to the control of a symmetric flexible system is investigated in this section. An internal and external decoupling scheme will be developed to simplify the complexity in controller design.

First consider the system of Eq. (25) with dimension $2n$ satisfying Eqs. (26) and (27). Let a transformation matrix P satisfy Eq. (7) [or Eq. (8)]. Then substitution of $q = P\omega$ into Eq. (25) yields

$$\begin{bmatrix} M_s & 0 \\ 0 & M_a \end{bmatrix} \begin{bmatrix} \ddot{\omega}_s \\ \ddot{\omega}_a \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & K_a \end{bmatrix} \begin{bmatrix} \omega_s \\ \omega_a \end{bmatrix} = 0 \tag{37}$$

Thereby, the system is separated into two uncoupled subsystems. This decoupling property is referred to as internal decoupling. For any value of λ, the transformed characteristic equation is of the form

$$|P^T K P - \lambda P^T M P| = |P^T|\,|P|\,|K - \lambda M| = 0 \tag{38}$$

Hence, the transformed eigenvalues are unchanged if matrix P is nonsingular, not necessarily orthogonal. (This type of transformation is not the same as that of similarity transformation of matrix.) The main advantage of this internal decoupling scheme is that the original eigenvalue problem can be separated into two eigenvalue problems of lower order, and thus it allows us to solve the eigenvalue problem with less computational effort.

In addition to the invariance of eigenvalues under transformation, the original characteristics of symmetric and antisymmetric modes can also be retained in each subsystem. This property is shown as follows. Let us consider the original symmetric and antisymmetric modes, V_s and V_a, respectively. They can be transformed into

$$\begin{aligned} P^{-1}V_s &= \begin{bmatrix} P_{11} & -BP_{22} \\ BP_{11} & P_{22} \end{bmatrix}^{-1} \begin{bmatrix} \phi_s \\ B\phi_s \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} P_{11}^{-1} & P_{11}^{-1}B \\ -P_{22}^{-1}B & P_{22}^{-1} \end{bmatrix} \begin{bmatrix} \phi_s \\ B\phi_s \end{bmatrix} = \begin{bmatrix} P_{11}^{-1}\phi_s \\ 0 \end{bmatrix} \end{aligned} \tag{39}$$

$$\begin{aligned} P^{-1}V_a &= \begin{bmatrix} P_{11} & -BP_{22} \\ BP_{11} & P_{22} \end{bmatrix}^{-1} \begin{bmatrix} \phi_a \\ -B\phi_a \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} P_{11}^{-1} & P_{11}^{-1}B \\ -P_{22}^{-1}B & P_{22}^{-1} \end{bmatrix} \begin{bmatrix} \phi_a \\ -B\phi_a \end{bmatrix} = \begin{bmatrix} 0 \\ -P_{22}^{-1}B\phi_a \end{bmatrix} \end{aligned} \tag{40}$$

Since the original eigenvalues λ_s and λ_a corresponding to symmetric and antisymmetric modes are invariant under transformation, the transformed eigenvalue problem becomes

$$(K_s - \lambda_s M_s)P_{11}^{-1}\phi_s = 0 \tag{41}$$

and

$$(K_a - \lambda_a M_a)P_{22}^{-1}B\phi_a = 0 \tag{42}$$

If we take $P_{11}=I$ and $P_{22}=-B$, the original mode shapes can be preserved. Therefore, the eigenstructure characteristics of the system are invariant under transformation, and this property is useful in controller design, e.g., eigenstructure assignment.

In fact, observing the constraint equations of P, Eqs. (7) and (8), the determination of transformation matrix P is not unique. For convenience, a set N (not exclusive) which includes 16 orthogonal matrices is given in Appendix B.

In addition to the symmetric flexible system, if the locations of actuators/sensors are allocated symmetrically, two groups of controllers/measurements can also be separated and obtained. This decoupling property is referred to as external decoupling.

Reconsidering the system, Eq. (37), when the actuators are added, the controlled system can be written as

$$\begin{bmatrix} M_s & 0 \\ 0 & M_a \end{bmatrix}\begin{bmatrix} \ddot{\omega}_s \\ \ddot{\omega}_a \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & K_a \end{bmatrix}\begin{bmatrix} \omega_s \\ \omega_a \end{bmatrix} = P^T D u \tag{43}$$

where D is the applied load distribution matrix and u the control input vector. Let the actuators be symmetrically allocated. Then it can be shown that the nonsquare matrix D satisfies

$$D = B_{2n} D B_{2m} \tag{44}$$

where the variable m denotes the number of actuators in pairs.

By using the results of Theorem 2 and choosing P and Q as orthogonal matrices, i.e.,

$$P^T = P^{-1} \tag{45}$$

$$Q^T = Q^{-1} \tag{46}$$

Eq. (43) can be written as

$$\begin{bmatrix} M_s & 0 \\ 0 & M_a \end{bmatrix}\begin{bmatrix} \ddot{\omega}_s \\ \ddot{\omega}_a \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & K_a \end{bmatrix}\begin{bmatrix} \omega_s \\ \omega_a \end{bmatrix} = P^T D Q Q^T u \tag{47}$$

$$\begin{bmatrix} M_s & 0 \\ 0 & M_a \end{bmatrix}\begin{bmatrix} \ddot{\omega}_s \\ \ddot{\omega}_a \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & K_a \end{bmatrix}\begin{bmatrix} \omega_s \\ \omega_a \end{bmatrix} = \begin{bmatrix} D_s & 0 \\ 0 & D_a \end{bmatrix}\begin{bmatrix} u_s \\ u_a \end{bmatrix} \tag{48}$$

where

$$\begin{bmatrix} D_s & 0 \\ 0 & D_a \end{bmatrix} = P^T D Q \qquad \begin{bmatrix} u_s \\ u_a \end{bmatrix} = Q^T u$$

It should be noted here that, since Q is chosen as the orthogonal matrix, the external decoupling of the original system does not need to use the inverse operation of matrix. Similarly, since the determination of transformation matrices P and Q is independent of the entries of applied load distribution matrix D, the output equation can also be decoupled into two subsystems if the sensors are symmetrically added but not necessarily collocated with actuators as discussed in the following.

Consider the case where the number of sensors is equal to that of actuators. The vector of measurement output is denoted by

$$z(t) = [z_d^T \quad z_v^T]^T \tag{49}$$

where the subscripts d and v represent the measurement output sensed from displacement and velocity sensors, respectively. The output equation can be written as

$$z_d(t) = C_d q = C_d P \begin{bmatrix} \omega_s \\ \omega_a \end{bmatrix} \tag{50}$$

and

$$z_v(t) = C_v \dot{q} = C_v P \begin{bmatrix} \dot{\omega}_s \\ \dot{\omega}_a \end{bmatrix} \tag{51}$$

where C_d and C_v are displacement and velocity sensor distribution matrices, respectively, and the transpose of both satisfies Eq. (44). Premultiplying Eqs. (50) and (51) by matrix Q^T yields

$$Q^T z_d(t) = Q^T C_d P \begin{bmatrix} \omega_s \\ \omega_a \end{bmatrix} = \begin{bmatrix} C_{ds} & 0 \\ 0 & C_{da} \end{bmatrix}\begin{bmatrix} \omega_s \\ \omega_a \end{bmatrix} \tag{52}$$

and

$$Q^T z_v(t) = Q^T C_v P \begin{bmatrix} \dot{\omega}_s \\ \dot{\omega}_a \end{bmatrix} = \begin{bmatrix} C_{vs} & 0 \\ 0 & C_{va} \end{bmatrix}\begin{bmatrix} \dot{\omega}_s \\ \dot{\omega}_a \end{bmatrix} \tag{53}$$

Rearrange the sensor output vector by defining

$$y = [y_s^T \quad y_a^T]^T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\begin{bmatrix} Q^T & 0 \\ 0 & Q^T \end{bmatrix} z \tag{54}$$

The output equation can be rewritten as

$$y_s(t) = \begin{bmatrix} C_{ds} & 0 \\ 0 & C_{vs} \end{bmatrix}\begin{bmatrix} \omega_s \\ \dot{\omega}_s \end{bmatrix} \tag{55}$$

and

$$y_a(t) = \begin{bmatrix} C_{da} & 0 \\ 0 & C_{va} \end{bmatrix}\begin{bmatrix} \omega_a \\ \dot{\omega}_a \end{bmatrix} \tag{56}$$

Thus, the whole controlled system is decoupled into two subsystems without using the inverse operation of the matrix if the transformation matrices P and Q not only satisfy constraint Eqs. (20) and (22) [or Eqs. (21) and (23)] but also are chosen as orthogonal matrices, e.g., they can be chosen from Appendix B.

Robustness Properties of Collocated Actuators and Sensors

Consider a direct output feedback based controller design in which the actuators and sensors are collocated. Based on the developed decoupling scheme, two robustness properties, FOCL stability and reliable control, are investigated.

FOCL Stability

Let the output feedback be defined by

$$\begin{aligned} u &= -Gz \\ &= -[G_d \quad G_v] z \end{aligned} \tag{57}$$

The equations of the FOCL system can be written as

$$M\ddot{q} + DG_v C_v \dot{q} + (K + DG_d C_d) q = 0 \tag{58}$$

According to the Kelvin-Tait-Chetaev stability theorem for mechanical systems,[14,15] the guarantee of the FOCL stability is achieved if

$$D = C_d^T = C_v^T \tag{59}$$

and

$$G_d = G_d^T > 0, \qquad G_v = G_v^T > 0 \qquad (\geq 0) \tag{60}$$

where the requirement of G_d to be positive definite takes into account the stability of rigid-body modes. This is the principle for the reduced-order controller design as proposed by Salm. To satisfy the condition of Eq. (60), he used an independent model space approach to design the reduced-order controller. But, when the internal and external decoupling scheme is ap-

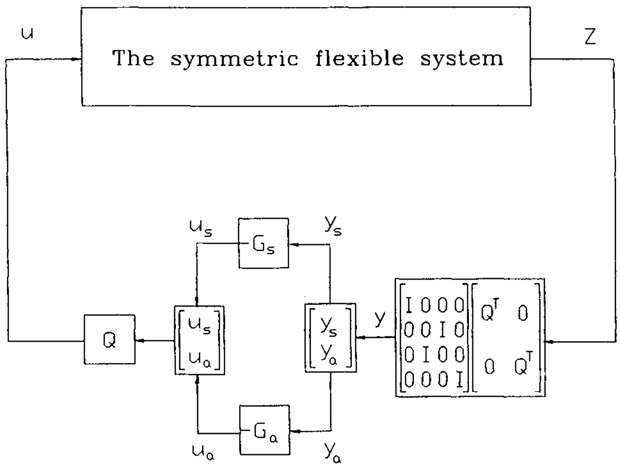

Fig. 1 Block diagram of direct output feedback via the internal and external decoupling scheme.

plied, any reduced-order controller design will guarantee the FOCL stability.

Reconsider the decoupled system of Eqs. (48), (55), and (56). The controller design of the system can be derived independently, as shown in Fig. 1. The symmetric subsystem is written as

$$M_s \ddot{\omega}_s + K_s \omega_s = D_s u_s \tag{61}$$

and

$$y_s = \begin{bmatrix} C_{ds} & 0 \\ 0 & C_{vs} \end{bmatrix} \begin{bmatrix} \omega_{xs} \\ \dot{\omega}_{xs} \end{bmatrix} \tag{62}$$

with linear output feedback

$$u_s = -G_s y_s = -[G_{ds} \quad G_{vs}] y_s \tag{63}$$

The antisymmetric subsystem is written as

$$M_a \ddot{\omega}_a + K_a \omega_a = D_a u_a \tag{64}$$

and

$$y_a = \begin{bmatrix} C_{da} & 0 \\ 0 & C_{va} \end{bmatrix} \begin{bmatrix} \omega_{xa} \\ \dot{\omega}_{xa} \end{bmatrix} \tag{65}$$

with linear output feedback

$$u_a = -G_a y_a = -[G_{da} \quad G_{va}] y_a \tag{66}$$

Let the number of actuators/sensors be equal to that of the subsystems so that each subsystem is controlled by one control input and observed by one sensor output. Under these circumstances, the feedback gains G_{ds}, G_{vs}, G_{da}, and G_{va} are, in effect, scalars.

The feedback gain matrix for the whole flexible system is then derived as

$$G = [G_d \quad G_v] = \left[Q \begin{bmatrix} G_{ds} & 0 \\ 0 & G_{da} \end{bmatrix} Q^T \quad Q \begin{bmatrix} G_{vs} & 0 \\ 0 & G_{va} \end{bmatrix} Q^T \right] \tag{67}$$

Since matrix Q is orthogonal, the condition of Eq. (60) is satisfied provided that G_{ds}, $G_{da} > 0$ and G_{vs}, $G_{va} \geq 0$ (i.e., FOCL stability of each subsystem is guaranteed). Thus, the FOCL stability of the whole system is guaranteed despite which approach is utilized in reduced-order controller design.

Reliable Control

Consider the case that only one of the actuators/sensors fails. Since actuators and sensors are collocated, the failure occurring here means that both actuator and sensor allocated in the same place break down. Reconsider the gain matrix of Eq. (67). Partitioning the matrix Q into two vectors yields

$$Q^T = [Q_1^T \quad Q_2^T] \tag{68}$$

Now suppose that the second actuator/sensor breaks down. The gain matrix for the failed system, which is now controlled by the first actuator/sensor, can be reduced as

$$G_1 = [G_{1d} \quad G_{1v}] = \left[Q_1 \begin{bmatrix} G_{ds} & 0 \\ 0 & G_{da} \end{bmatrix} Q_1^T \quad Q_1 \begin{bmatrix} G_{vs} & 0 \\ 0 & G_{va} \end{bmatrix} Q_1^T \right] \tag{69}$$

Thus, using the same deduction as earlier, it is claimed that the FOCL stability of the failed system is preserved. In the case of the first actuator/sensor breakdown, a similar result can be obtained. Integrally, any one of the actuators/sensors can be taken as a backup for the other. The main advantage of this type of reliable control is that there is no need for failure detection and reconfiguration of controller design.

Optimal Modal Space Approach in Controller Design

After the symmetric flexible system is decoupled into two subsystems, the control algorithm for each of them can be derived independently and then be joined together to obtain the system controller under the guarantee of FOCL stability. The optimal modal space approach is adopted to provide active stiffness and damping for each subsystem. Reconsider the symmetric subsystem of Eqs. (61) and (62). Let a transformation matrix ϕ_s be the matrix of eigenvectors of K_s normalized with respect to the mass matrix M_s so that

$$\phi_s^T M_s \phi_s = I \tag{70}$$

and

$$\phi_s^T K_s \phi_s = \mathrm{diag}[\Omega_{si}^2] \tag{71}$$

where Ω_{si}^2 are the eigenvalues of the matrix K_s. Substitution of $\omega_s = \phi_s \eta_s$ into Eq. (61) and premultiplying by ϕ_s^T yields

$$\ddot{\eta}_s + \{\mathrm{diag}[\Omega_{si}^2]\} \eta_s = \phi_s^T D_s u_s \tag{72}$$

Then partition this modal equation (72) into the t controlled modal coordinates and the $n-t$ residual modal coordinates. This yields

$$\begin{bmatrix} \ddot{\eta}_{sc} \\ \ddot{\eta}_{sr} \end{bmatrix} + \begin{bmatrix} \mathrm{diag}[\Omega_{sci}^2] & 0 \\ 0 & \mathrm{diag}[\Omega_{sri}^2] \end{bmatrix} \begin{bmatrix} \eta_{sc} \\ \eta_{sr} \end{bmatrix} = \begin{bmatrix} \phi_{sc}^T D_s \\ \phi_{sr}^T D_s \end{bmatrix} u_s \tag{73}$$

The reduced-order controlled subsystem is given by

$$\ddot{\eta}_{sc} + \{\mathrm{diag}[\Omega_{sci}^2]\} \eta_{sc} = \phi_{sc}^T D_s u_s \tag{74}$$

The vector of sensor output is also reduced to

$$y_s = \begin{bmatrix} C_{ds} & 0 \\ 0 & C_{vs} \end{bmatrix} \begin{bmatrix} \phi_{sc} & 0 \\ 0 & \phi_{sc} \end{bmatrix} \begin{bmatrix} \eta_{sc} \\ \dot{\eta}_{sc} \end{bmatrix} \tag{75}$$

Based on Eqs. (74) and (75), the optimal state feedback control system is developed. The state space realization of Eq. (74) is given by

$$\begin{bmatrix} \dot{\eta}_{sc} \\ \ddot{\eta}_{sc} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathrm{diag}[\Omega_{sci}^2] & 0 \end{bmatrix} \begin{bmatrix} \eta_{sc} \\ \dot{\eta}_{sc} \end{bmatrix} + \begin{bmatrix} 0 \\ \phi_{sc}^T D_s \end{bmatrix} u_s \tag{76}$$

For simplicity, define

$$A = \begin{bmatrix} 0 & I \\ -\mathrm{diag}[\zeta^2_{sci}] & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ \phi^T_{sc} D_s \end{bmatrix}, \quad s = \begin{bmatrix} \eta_{sc} \\ \dot{\eta}_{sc} \end{bmatrix}$$

so that Eq. (76) becomes

$$\dot{s} = As + Hu_s \tag{77}$$

The linear output feedback can be rewritten as

$$\begin{aligned} u_s &= -G_s y_s \\ &= -G_s \begin{bmatrix} C_{ds} & 0 \\ 0 & C_{vs} \end{bmatrix} \begin{bmatrix} \phi_{sc} & 0 \\ 0 & \phi_{sc} \end{bmatrix} s \end{aligned} \tag{78}$$

Now define a cost function

$$J = \int_0^{\infty} (s^T Q s + u_s^T R u_s)\, \mathrm{d}t \tag{79}$$

where the weighting matrix Q is at least symmetric, positive semidefinite and $R = \mathrm{diag}[r_i]$ is the positive definite. They govern the relative importance of modal coordinates η_{sci} and the control vector u_s. In addition, let a matrix P satisfy the steady state Riccati equation given by

$$PA + A^T P - PHR^{-1}H^T P + Q = 0 \tag{80}$$

Thus, the minimization of Eq. (79) is achieved if the gain matrix G_s is computed from

$$G_s = [G_{ds} \quad G_{vs}] = \left[F_d (C_{ds}\phi_{sc})^{\#} \quad F_v (C_{vs}\phi_{sc})^{\#}\right] \tag{81}$$

where $[F_d \quad F_v] = R^{-1}H^T P$ and $(\ldots)^{\#}$ denotes the Moore-Penrose pseudoinverse (see Appendix C). From a practical point of view, the sensor distribution matrices $C_{ds}\phi_{sc}$ and $C_{vs}\phi_{sc}$ used

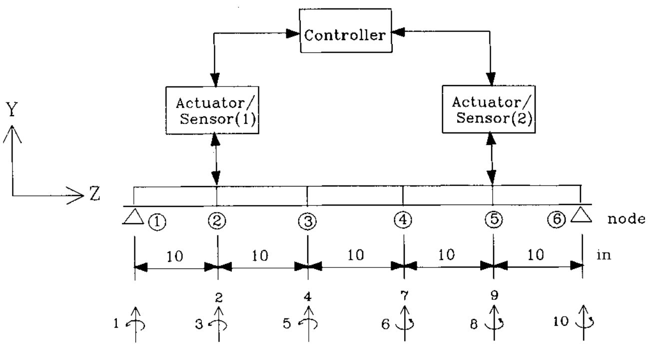

Fig. 2 Controlled uniform simply supported beam.

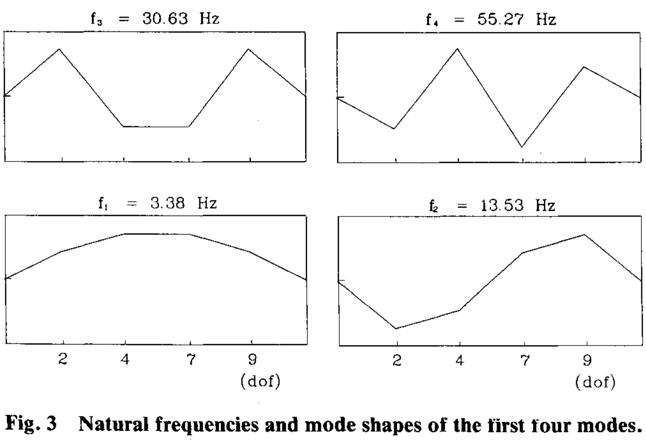

Fig. 3 Natural frequencies and mode shapes of the first four modes.

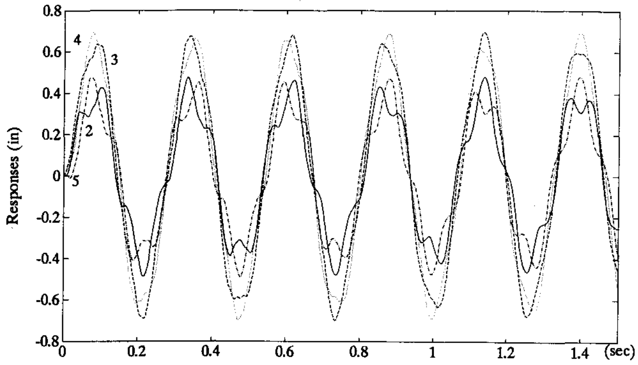

Fig. 4 Responses of the beam system at nodes 2–5 without control.

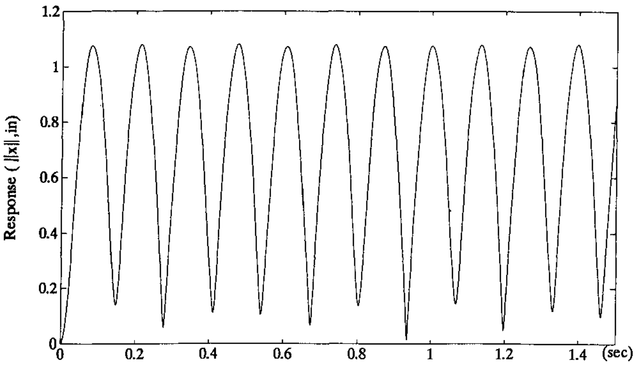

Fig. 5 Global response of the beam system without control.

here are always of full rank if the locations of sensors do not coincide with the vibration nodes. It is noted here that, although the spillover effect exists in the subsystem itself, it is prevented from occurring between those two subsystems.

In view of the preceding, the advantages of the controller design via ad hoc internal and external decoupling schemes are listed as follows.

1) There is less computational effort in solving the eigenvalue problem from the order of $2n$ to the one of n.

2) There is less computational effort in solving the Riccati equation from the order of $2t$ to the one of t.

3) Less computer storage is required.

4) Less computational effort is required in computing the inverse operation of the matrix.

5) The approximation error due to the Moore-Penrose pseudoinverse operation of the matrix is lower, e.g., the approximation by the pseudoinverse operation from matrix $2 \times 2t$ matrix $1 \times t$.

6) The scheme is more flexible in controller design, e.g., each subsystem is controlled by its own control algorithm.

7) The scheme is more suitable for implementation, e.g., execution by different computers.

8) Spillover effect is prevented from happening between subsystems.

Simulation Results and Discussions

A numerical example considered in this study is a uniform simply supported beam, as shown in Fig. 2. The beam is made of aluminum with a length of 50 in., and a rectangular cross section of width 1.5 in. and thickness 0.25 in. The modulus of elasticity is 1.0×10^7 lbf/in.^{2}; and the mass per unit length is 6.763×10^{-4} lbf·s^2/in.^{2}. A finite element model of this beam system is composed of 5 elements having a total of 10 degrees of freedom (translation and rotation at nodes 2–5, and rotation at both ends). According to the physical properties of the beam, the first four transverse natural frequencies and mode shapes [represented by degrees of freedom (DOF) 2, 4, 7, 9] are shown in Fig. 3. Obviously, the set of mode shapes is composed of symmetric (e.g., $f = 3.38$, 30.63 Hz) and antisymmetric (e.g., $f = 13.53$, 55.27 Hz) modes.

An initial impulse of 0.03-s duration and 10 lbf is applied at 20 in. from the left end (at node 3). The collocated sensors and actuators are symmetrically located 10 in. from both ends (at nodes 2 and 5). The transient simulation of the beam system is performed by Wilson-θ method with the Wilson's coefficient 1.37 and the integration time step 0.001 s. The transverse responses at nodes 2–5 to the initial impulse just described without control are shown in Fig. 4. To evaluate the global response of the beam system, the norm of the response is used in the following analysis. The response vector $\{x(t)\}$ contains only the translations of the elements, and its norm is defined as

$$\|x\| = \sqrt{\{x(t)\}^T\{x(t)\}}$$

The global response to the initial impulse without control is again shown in Fig. 5.

Consider the global response of the beam system under control. The optimal modal space approach discussed earlier is applied to obtain the output feedback gain. By using the decoupling scheme, the whole system is decoupled into two subsystems, i.e., symmetric and antisymmetric subsystems. For the symmetric subsystem, the controlled mode is the first ($f = 3.38$ Hz) mode; and for the antisymmetric subsystem, the controlled mode is the second ($f = 13.53$ Hz) flexible mode. The weighting matrix for both subsystems is chosen as

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$$

and $r = 0.1$. Figure 6 shows the global response of the beam system under this output feedback control. It is observed that the initial disturbance is damped out after 0.6 s. The control force history is shown in Fig. 7.

Consider the case that one of the actuators/sensors fails abruptly during control. In the process of simulation, the failure of the second actuator/sensor (at node 5) is assumed to have occurred at time $t = 0.5$ s, and at the same time the beam system is also disturbed by an impulse of 0.03-s duration and 10 lbf at node 3. The global response of this case is shown in Fig. 8. In the case where the failure of the first actuator/sensor occurred, we obtain a similar result, as also shown in Fig. 8. Both of the failed systems are still stabilized without failure detection and reconfiguration of controller design.

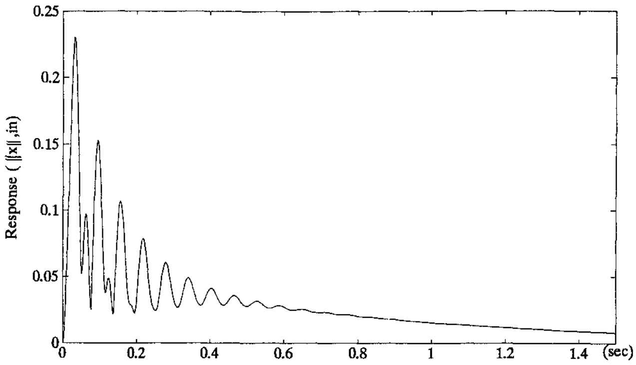

Fig. 6 Global response of the beam system under control.

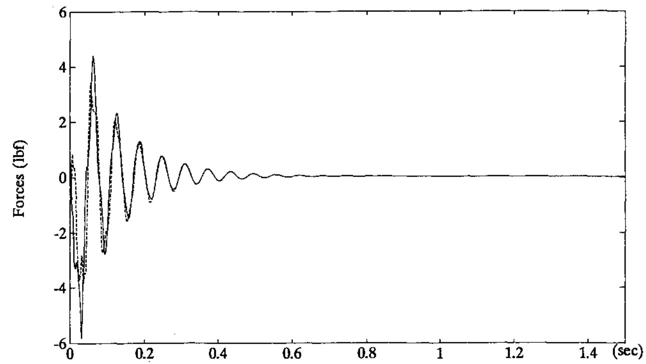

Fig. 7 Force histories of the first (——) and second (-----) actuators.

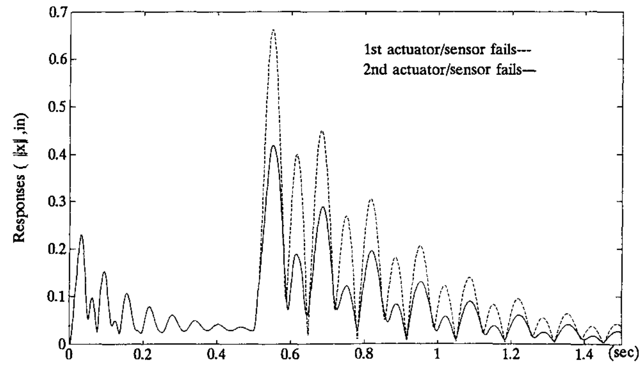

Fig. 8 Global responses of the beam system under control (with failure).

Conclusion

In this paper, the advantages that come from the use of symmetry are introduced and investigated in the vibration analysis and control of a symmetric flexible system. The formulation of the symmetric matrix with reflection symmetry is presented to describe the symmetry property of physical systems. This results in a great saving of effort in vibration analysis. Two important theorems used to decouple the original matrix into two uncoupled submatrices are further presented. Consider the symmetric flexible system with actuators/sensors being symmetrically allocated. Those theorems are applied to internally and externally decouple the whole system into the subsystems. The control algorithm can be derived independently so that it reduces much computational complexity in controller design and improves the feasibility in implementation. Furthermore, consider the controller design with direct output feedback and with collocation of sensors and actuators. Any reduced-order controller design would guarantee the FOCL stability and, simultaneously, carry the implication of a passive type of reliable control.

Appendix A: Backward Identity Matrix

The backward identity matrix (see also Horn and Johnson[16]) is represented as

$$B = \begin{bmatrix} & & & 1 \\ & 0 & & \\ & & \cdot^{\cdot^{\cdot}} & \\ & & 0 & \\ 1 & & & \end{bmatrix} \tag{A1}$$

which possesses the properties $B = B^T$ and $B^2 = I$.

Appendix B: Set of Transformation Matrix

$$
N = \left\{
\begin{bmatrix} 1 & & & & & 1 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & 1 & & \\ & & -1 & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ -1 & & & & & 1 \end{bmatrix},
\begin{bmatrix} 1 & & & & & 1 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & 1 & & \\ & & 1 & -1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ 1 & & & & & -1 \end{bmatrix},
\begin{bmatrix} 1 & & & & & -1 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & -1 & & \\ & & 1 & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ 1 & & & & & 1 \end{bmatrix}
\right.
$$

$$
\begin{bmatrix} -1 & & & & & 1 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & -1 & 1 & & \\ & & 1 & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ 1 & & & & & 1 \end{bmatrix},
\begin{bmatrix} 1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & 1 \\ & & -1 & & & 1 \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & \\ -1 & & & 1 & & \end{bmatrix},
\begin{bmatrix} 1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & 1 \\ & & 1 & & & -1 \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & \\ 1 & & & -1 & & \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & & & -1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & -1 \\ & & 1 & & & 1 \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & \\ 1 & & & 1 & & \end{bmatrix},
\begin{bmatrix} -1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & -1 & & & 1 \\ & & 1 & & & 1 \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & \\ 1 & & & 1 & & \end{bmatrix},
\begin{bmatrix} & & 1 & & & 1 \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & \\ 1 & & & 1 & & \\ -1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & -1 & & & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} & & 1 & & & 1 \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & \\ 1 & & & 1 & & \\ 1 & & & -1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & -1 \end{bmatrix},
\begin{bmatrix} & & 1 & & & -1 \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & \\ 1 & & & -1 & & \\ 1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & 1 \end{bmatrix},
\begin{bmatrix} & & -1 & & & 1 \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & \\ -1 & & & 1 & & \\ 1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} & & 1 & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ 1 & & & & & 1 \\ -1 & & & & & 1 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & -1 & 1 & & \end{bmatrix},
\begin{bmatrix} & & 1 & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ 1 & & & & & 1 \\ 1 & & & & & -1 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & -1 & & \end{bmatrix},
\begin{bmatrix} & & 1 & -1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ 1 & & & & & -1 \\ 1 & & & & & 1 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & 1 & & \end{bmatrix}
$$

$$
\left.
\begin{bmatrix} & & -1 & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ -1 & & & & & 1 \\ 1 & & & & & 1 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & 1 & & \end{bmatrix}
\right\}
\tag{B1}
$$

Appendix C: Moore-Penrose Pseudoinverse

The Moore-Penrose pseudoinverse for a rectangular column-regular matrix B (linear independent columns, B^TB nonsingular) is

$$(B^TB)^{-1}B^T \tag{C1}$$

and for a rectangular row-regular matrix C (linear independent columns, C^TC nonsingular) is

$$C^T(CC^T)^{-1} \tag{C2}$$

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